11.1. Gromov–Hausdorff distance.

(a) Show that if X, Y are two compact metric spaces with $d_{\mathrm{GH}}(X, Y) = 0$, then X and Y are isometric.

(b) Consider the two bounded, complete geodesic metric spaces X, Y defined as follows: X is a tree with one central vertex and edges of length $1-\frac{1}{2}, 1-\frac{1}{3}, 1-\frac{1}{4}, \ldots$ attached to it, and Y is constructed similarly, but with an additional segment of length 1. Show that $d_{\mathrm{GH}}(X, Y) = 0$ despite X and Y being non-isometric.

Solution. (a) See for example p. 73 in Bridson–Haefliger, Metric Spaces of Non-Positive Curvature, Springer 1999, or p. 259 in Burago–Burago–Ivanov, A Course in Metric Geometry, AMS 2001.

(b) Let Z be the tree with one central vertex and a countably infinite number of edges of length 1. For $i \geq 2$, let X_i, Y_i denote the edges of X, Y, respectively, of length $1 - \frac{1}{i}$, and let Y_∞ denote the additional edge of length 1. Both X and Y embed isometrically into Z, and one can match up X_i with Y_i for $i = 2, \ldots, k-1$, then X_k with Y_∞, and X_{j+1} with Y_j for $j = k, k+1, \ldots$, to see that $d_{\mathrm{GH}}(X, Y) \leq 1/k$.

□

11.2. Gromov–Hausdorff limits. Suppose that a sequence of metric spaces X_i converges in the Gromov–Hausdorff distance to a complete metric space X. Prove the first three assertions in Remark 6.19 (recall the definitions from Section 1.6). Hint for (1): Show that a complete metric space X is a length space if and only if for every pair of points $x, y \in X$ and $\epsilon > 0$ there exists an approximate midpoint $z \in X$ with $d(x, z), d(z, y) \leq \frac{1}{2} d(x, y) + \epsilon$.

Solution. See again p. 73 in Bridson–Haefliger. □

11.3. Increase/collapse of dimension.

(a) Find a sequence of closed Riemannian surfaces that converges in the Gromov–Hausdorff distance to the unit cube $C := [0,1]^3 \subset \mathbb{R}^3$ endowed with the l_1-distance $d_1(x, y) := \|x - y\|_1 = \sum_{n=1}^{3} |x_n - y_n|$.

(b) Find a sequence of Riemannian 3-spheres (S^3, g_k) that converges in the Gromov–Hausdorff distance to the 2-sphere $S^2(\frac{1}{2}) = \mathbb{M}^2_4$ of constant curvature 4.

Solution. (a) For every integer $k \geq 1$, let $S_k \subset C$ be the 1-skeleton of the canonical subdivison of C into cubes of edge length $1/k$. Now take the boundary of the tubular $1/(100k)$-neighborhood (say) of S_k in $\mathbb{R}^3$ and deform it slightly to a smooth closed surface M_k.

(b) Let $\pi \colon S^3 \to \mathbb{C}P^1$ be the Hopf fibration. Rescale the metric of S^3 along the Hopf fibers so they have length $2\pi/k$ with respect to g_k, and note that $\mathbb{C}P^1$ with the Fubini–Study metric is isometric to $S^2(\frac{1}{2}) = \mathbb{M}^2_4$ (in Example 4.9, $\langle \bar{x}, i\bar{y} \rangle^2 = 1$ for $n = 1$).

$\square$